GEOMETRY

Dr. Marius Ghergu

School of Mathematics and Statistics

University College Dublin

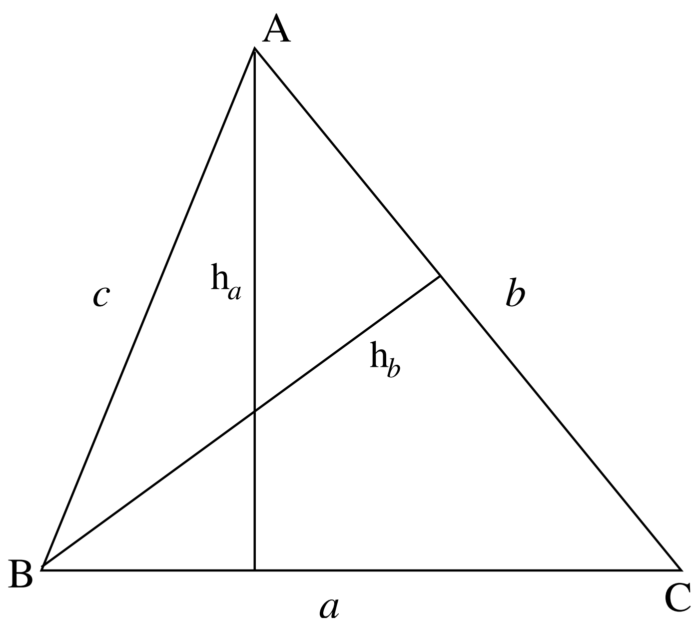
Standard notations for a triangle ABC

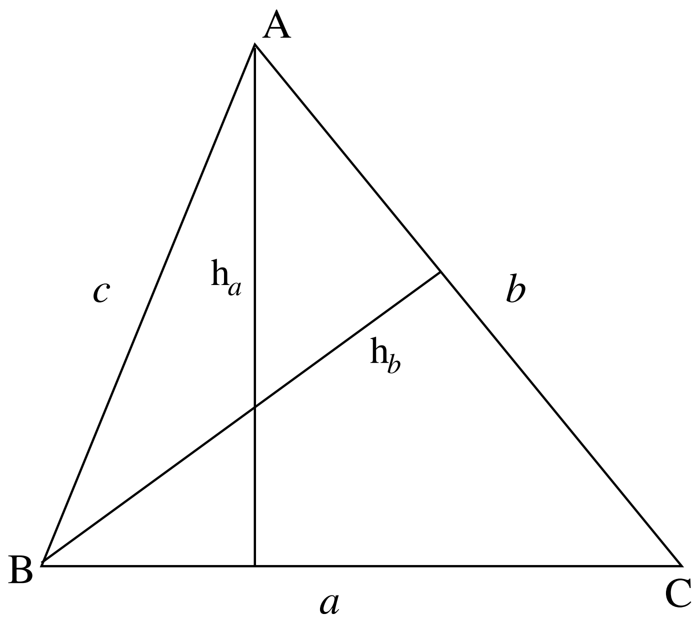
$$a = BC, \quad b = CA, \quad c = AB$$

$$h_a = \text{ the altitude from } A$$

$$h_b = \text{ the altitude from } B$$

$$h_c = \text{ the altitude from } C$$





Area of a triangle ABC is given by

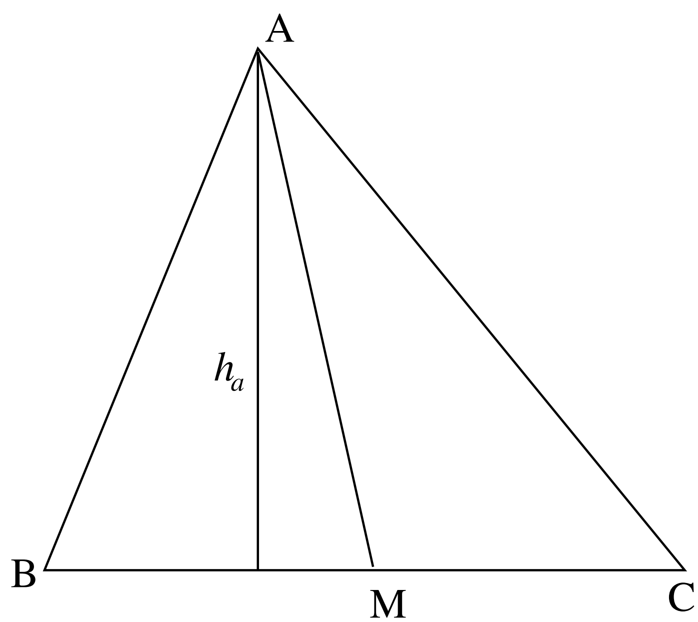
$$[ABC] = \frac{BC \cdot h_a}{2} = \frac{CA \cdot h_b}{2} = \frac{AB \cdot h_c}{2}$$

$$[ABC] = \frac{AB \cdot AC \cdot \sin \angle BAC}{2}$$

Proposition. The median of a triangle divides it into two triangles of the same area.

Proof. Indeed, if M is the midpoint of BC then

$$[ABM] = \frac{BM \cdot h_a}{2} = \frac{CM \cdot h_a}{2} = [ACM]$$

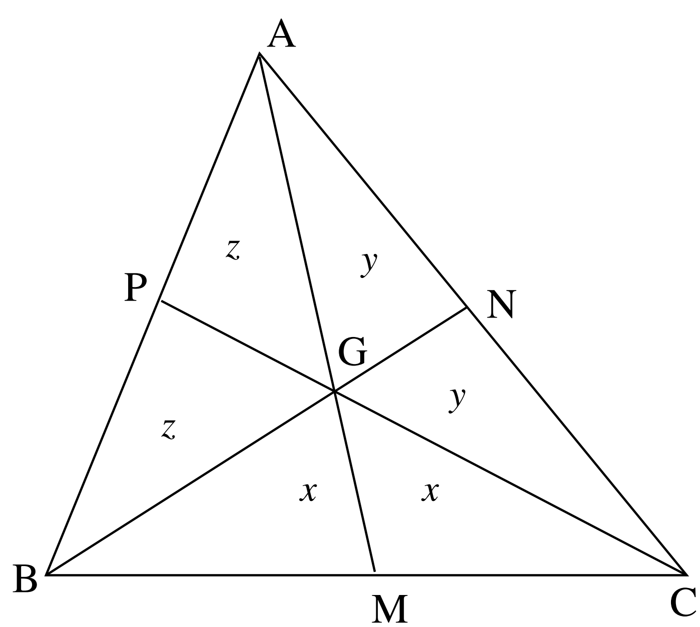


Problem 1. Let G be the centroid of a triangle $[ABC]$ (that is, the point of intersection of all its three medians). Then

$$[GAB] = [GBC] = [GCA].$$

Solution. Let M, N, P be the midpoints of BC, CA and AB respectively. Denote

$$[GMB] = x, \quad [GNA] = y, \quad [GPB] = z.$$



Note that GM is median in triangle GBC so

$$[GMC] = [GMB] = x.$$

Similarly $[GNC] = [GNA] = y$ and $[GPA] = [GPB] = z$.

Now $[ABM] = [ACM]$ implies $2z + x = 2y + x$ so $z = y$.

From $[BNC] = [BNA]$ we obtain $x = z$, so $x = y = z$

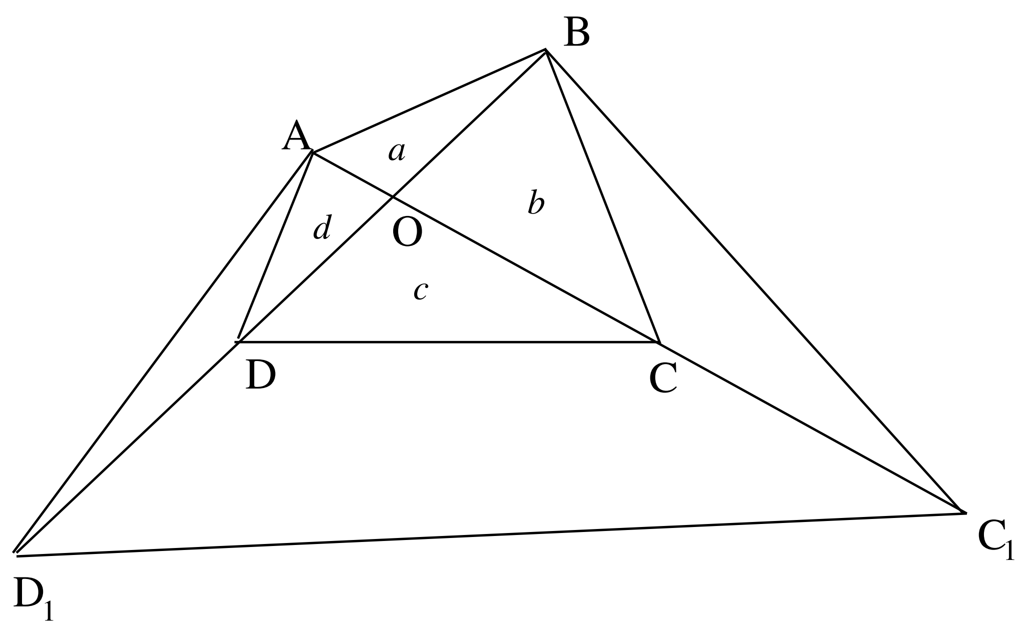
Problem 2. Let $ABCD$ be a convex quadrilateral. On the line AC we take the point C_1 such that $CA = CC_1$ and on the line BD we take the point D_1 such that $BD = DD_1$. Prove

$$[ABC_1D_1] = 4[ABCD].$$

Solution. Let O be the intersection of the diagonals AC and BD and denote

$$a = [AOB], \quad b = [BOC], \quad c = [COD], \quad d = [DOA].$$

Remark that AD is a median in triangle ABD_1 so



$$[ADD_1] = [ADB] = a + d.$$

BC is median in triangle ABC_1 so

$$[BCC_1] = [ABC] = a + b,$$

DC is median in triangle ADC_1 so

$$[DCC_1] = [ADC] = c + d.$$

Finally, C_1D is median in triangle BC_1D_1 so

$$[DD_1C_1] = [BDC_1] = a + 2(b + c) + d.$$

Now

$$[ABC_1D_1] = 4(a + b + c + d) = 4[ABCD].$$

Problem 3. Let M be a point inside a triangle ABC whose altitudes are h_a, h_b and h_c. Denote by d_a, d_b and d_c the distances from M to the sides BC, CA and AB respectively. Prove that

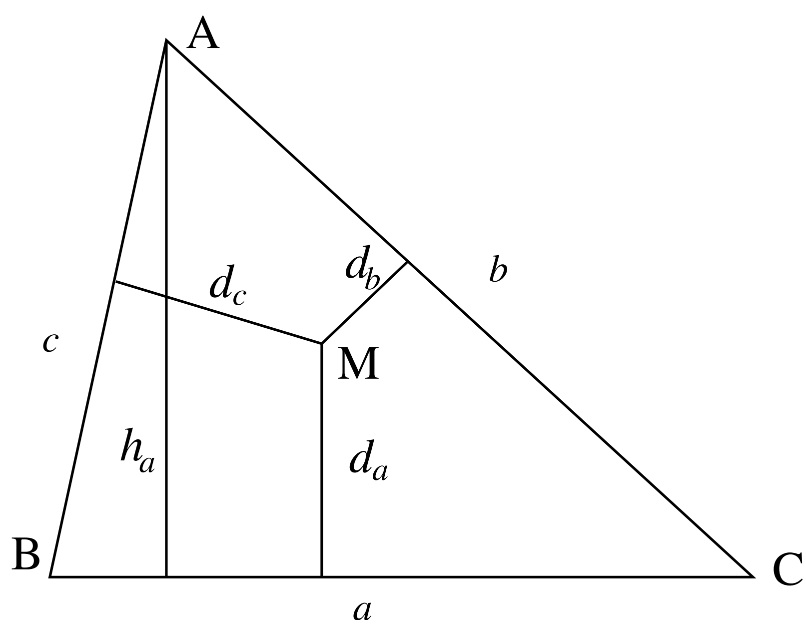
$$\min\{h_a, h_b, h_c\} \le d_a + d_b + d_c \le \max\{h_a, h_b, h_c\}.$$

Solution. Assume $a \ge b \ge c$. Since

$$2[ABC] = ah_a = bh_b = ch_c$$

it follows that

$$h_a \le h_b \le h_c.$$



$$2[ABC] = [BMC] + [2CMA] + 2[AMB]$$

$$2[ABC] = ad_a + bd_b + cd_c \ge c(d_a + d_b + d_c)$$

$$ch_c \ge c(d_a + d_b + d_c).$$

Hence

$$h_c \geq d_a + d_b + d_c.$$

Similarly we have

$$ah_a = 2[ABC] = ad_a + bd_b + cd_c \leq a(d_a + d_b + d_c)$$

which yields

$$h_a \leq d_a + d_b + d_c.$$

Let ABC and $A'B'C'$ be two similar triangles, that is,

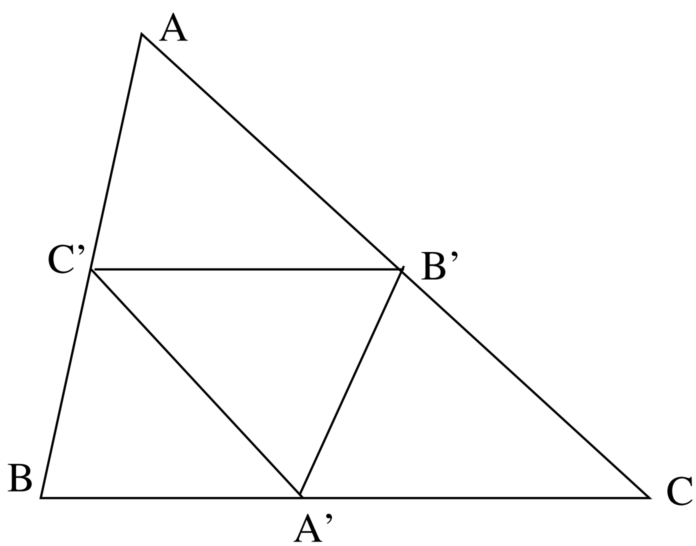
$$\frac{A'B'}{AB} = \frac{A'C'}{AC} = \frac{B'C'}{BC} = \text{ratio of similarity}$$

Then

$$\frac{[A'B'C']}{[ABC]} = \left(\frac{A'B'}{AB}\right)^2 = \left(\frac{B'C'}{BC}\right)^2 = \left(\frac{C'A'}{CA}\right)^2.$$

Proposition. The ratio of areas of two similar triangles equals the square of ratio of similarity.

Example. Consider the median triangle $A'B'C'$ of a triangle ABC (A', B' and C' are the midpoints of the sides of triangle ABC). The similarity ratio is



$$\frac{A'B'}{AB} = \frac{A'C'}{AC} = \frac{B'C'}{BC} = \frac{1}{2}$$

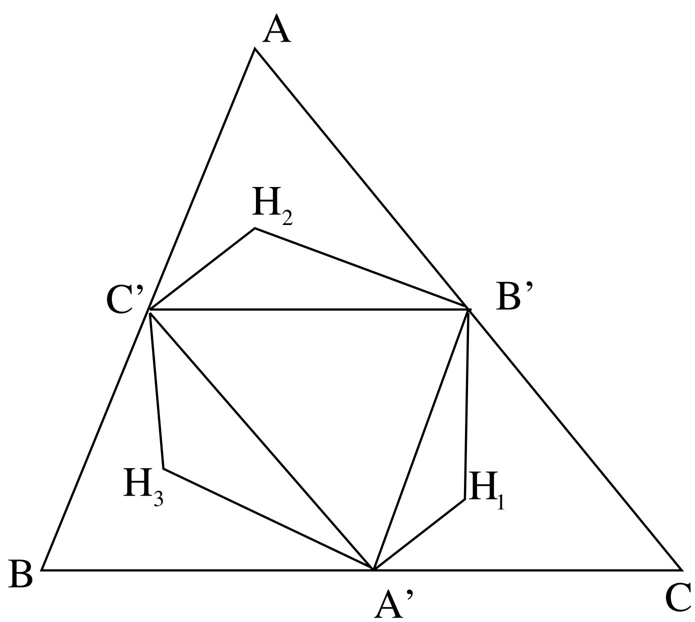
so

$$\frac{[A'B'C']}{[ABC]} = \left(\frac{A'B'}{AB}\right)^2 = \frac{1}{4} \quad \text{that is,} \quad [A'B'C'] = \frac{1}{4}[ABC].$$

Problem 4. Let $A'B'C'$ be the median triangle of ABC and denote by H_1, H_2 and H_3 the orthocentres of triangles $CA'B'$, $AB'C'$ and $BC'A'$ respectively.
Prove that

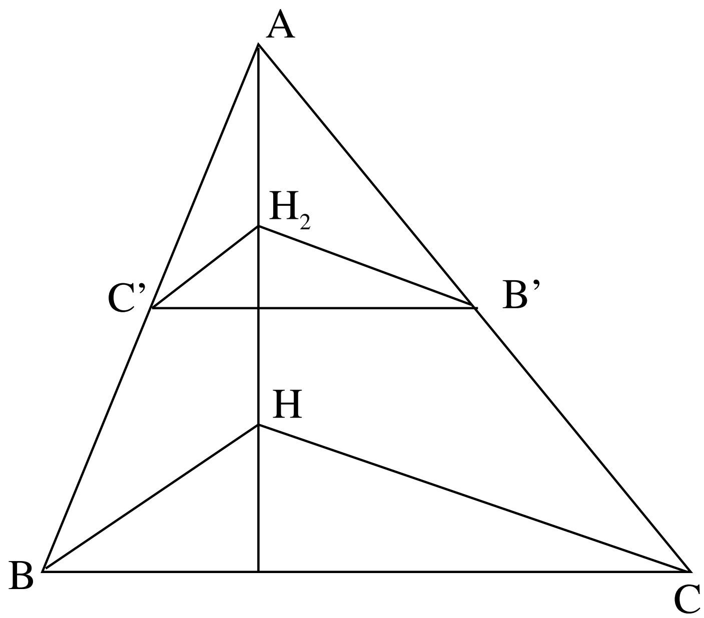
$$[A'H_1B'H_2C'H_3] = \frac{1}{2}[ABC].$$

Solution.



First remark that $A'B'C'$ and ABC are similar triangles with the similarity ratio $B'C' : BC = 1 : 2$. Therefore

$$[A'B'C'] = \frac{1}{4}[ABC].$$



Let H be the orthocentre of ABC. Then A, H_2 and H are on the same line. Also triangles $H_2C'B'$ and HBC are similar with the same similarity ratio, thus

$$[H_2B'C'] = \frac{1}{4}[HBC].$$

In the same way we obtain

$$[H_1A'B'] = \frac{1}{4}[HAB] \quad \text{and} \quad [H_3C'A'] = \frac{1}{4}[HCA].$$

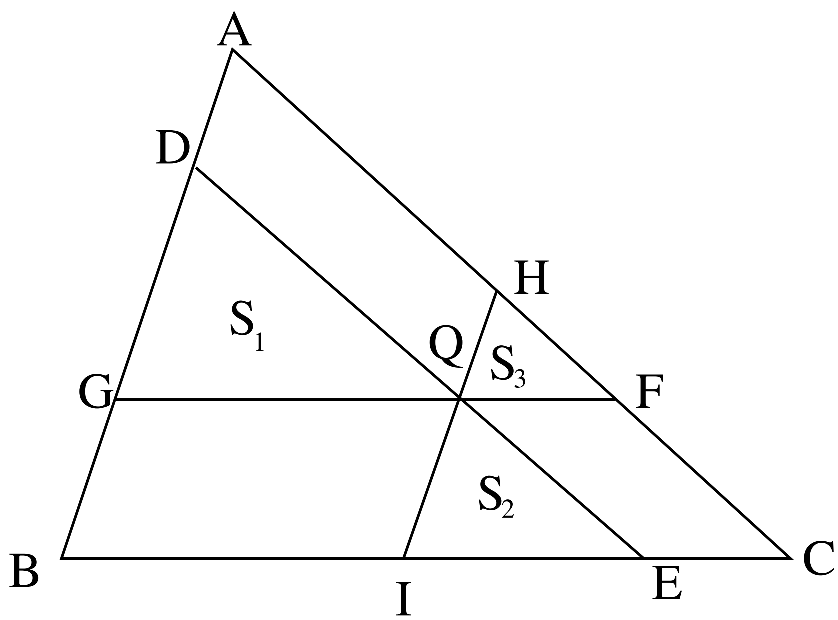
We now obtain

$$\begin{aligned}[A'H_1B'H_2C'H_3] &= [A'B'C'] + [H_1A'B'] + [H_2B'C'] + [H_3C'A'] \\ &= \frac{1}{4}[ABC] + \frac{[HAB] + [HBC] + [HCA]}{4} \\ &= \frac{1}{4}[ABC] + \frac{1}{4}[ABC] = \frac{1}{2}[ABC].\end{aligned}$$

Problem 5. Let Q be a point inside a triangle ABC. Three lines pass through Q and are parallel with the sides of the triangle. These lines divide the initial triangle into six parts, three of which are triangles of areas S_1, S_2 and S_3. Prove that

$$\sqrt{[ABC]} = \sqrt{S_1} + \sqrt{S_2} + \sqrt{S_3}.$$

Solution.



Let D, E, F, G, H, I be the points of intersection between the three lines and the sides of the triangle.

Then triangles DGQ, HQF, QIE and ABC are similar so

$$\frac{S_1}{[ABC]} = \left(\frac{GQ}{BC}\right)^2 = \left(\frac{BI}{BC}\right)^2$$

Similarly

$$\frac{S_2}{[ABC]} = \left(\frac{IE}{BC}\right)^2, \quad \frac{S_3}{[ABC]} = \left(\frac{QF}{BC}\right)^2 = \left(\frac{CE}{BC}\right)^2.$$

Then

$$\sqrt{\frac{S_1}{[ABC]}} + \sqrt{\frac{S_2}{[ABC]}} + \sqrt{\frac{S_3}{[ABC]}} = \frac{BI}{BC} + \frac{IE}{BC} + \frac{EC}{BC} = 1.$$

This yields

$$\sqrt{[ABC]} = \sqrt{S_1} + \sqrt{S_2} + \sqrt{S_3}.$$

Homework

1. Let ABC be a triangle. On the line BC, beyond the point C we take the point A' such that $BC = CA'$. On the line CA beyond the point A we take the point B' such that $AC = AB'$. On the line AB, beyond the point B we take the point C' such that $AB = BC'$. Prove that

$$[A'B'C'] = 7[ABC].$$

2. Let $ABCD$ be a quadrilateral. On the line AB, beyond the point B we take the point A' such that $AB = BA'$. On the line BC beyond the point C we take the point B' such that $BC = CB'$. On the line CD beyond the point D we take the point C' such that $CD = DC'$. On the line DA beyond the point A we take the point D' such that $DA = AD'$. Prove that

$$[A'B'C'D'] = 5[ABCD].$$

3. Let G be the centroid of triangle ABC. Denote by G_1, G_2 and G_3 the centroids of triangles ABG, BCG and CAG. Prove that

$$[G_1G_2G_3] = \frac{1}{9}[ABC].$$

Hint: Let T be the midpoint of AG. Then G_1 belongs to the line BT and divides it in the ration 2:1. Similarly G_3 belongs to the line CT and divides it in the ratio 2:1. Deduce that G_1G_3 is parallel to

BC and $G_1G_3 = \frac{1}{3}BC$. Using this argument, deduce that triangles $G_1G_2G_3$ and ABC are similar with ratio of similarity of 1/3.

4. Let A', B' and C' be the midpoints of the sides BC, CA and AB of triangle ABC. Denote by G_1, G_2 and G_3 the centroids of triangles $AB'C'$, $BA'C'$ and $CA'B'$. Prove that

$$[A'G_2B'G_1C'G_3] = \frac{1}{2}[ABC].$$

5. Let M be a point inside a triangle ABC such that

$$[MAB] = [MBC] = [MCA].$$

Prove that M is the centroid of the triangle ABC.